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A NUMERICAL SOLUTION OF A FREE-BOUNDARY PROBLEM ON THE UNSATURATED FLOW OF LIQUIDS IN POROUS MEDIA

A. KHARAB
University of Petroleum & Minerals, Department of Mathematics, Dhahran 31261, Saudi Arabia

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Abstract—A numerical approximation is used to present the solution of a free-boundary problem in one space dimension, arising in the filtration of a fluid through a porous medium. The numerical scheme presented combines the use of finite elements and a predictor–corrector method. A numerical algorithm for finding the approximate solution and results of numerical calculations are given.

1. INTRODUCTION

Free-boundary problems occur in many areas and are solved routinely with the computer. Many numerical methods have been used and new or modified ones are being developed (see Refs [1, 2] and the references given therein). One of these methods being commonly used in the one-dimensional problem, is boundary fixing. This method consists of changing the free-boundary problem to one with a fixed boundary by use of the so-called Landau transformation. By using this method one can avoid the difficulties arising from having an unknown domain for the differential equation; it also yields numerous numerical methods which are simple and efficient to use. The disadvantage of it is the nonlinear structure of the transformed governing differential equation.

This paper describes the application of this method to find the numerical solution of a free-boundary problem in one space dimension, arising in the filtration of a compressible fluid through a porous medium. This problem has been studied from the theoretical point of view in a more general case by DiBenedetto [3], where the regularity of weak solutions is discussed as well as the proof of new results concerning regularity at the boundary. The governing equation of the problem that we will derive below depends on the so-called retention function which describes the water content in the soil at a given pressure. This function is of great interest and has been intensively studied in soil science. There are many mathematical expressions in the literature for soil–water retention curves [4]. The case treated here is that of a quadratic retention function which is commonly used in soil science.

In this paper we will first state the governing equation and the boundary conditions for the mathematical model. In Section 3 the free-boundary problem is transformed to one with fixed boundary and a weak formulation of the problem is stated. In Sections 4–6 a numerical scheme using finite elements and a predictor–corrector method is described and numerical results are given with a conclusion.

2. THE MATHEMATICAL MODEL

The problem we are dealing with arises from filtration of a fluid through porous media. We consider a homogeneous porous medium which consists of a large slab so that physical properties can be determined by the single variable y and which is assumed to be saturated with water to a certain depth. As time progresses the liquid will flow toward areas of lower pressure. The flow will continue as long as a sufficiently high saturation is maintained. Our problem is to describe the pressure $U(y, \tau)$ at a given time τ and to determine the depth to which the fluid has penetrated.

Using Darcy's law and the continuity equation [5] we derive the governing equation of our problem:

$$\frac{\partial R(U)}{\partial \tau} = \frac{\partial^2 U}{\partial y^2}, \tag{1}$$

where

$$R(U) = \phi S; \tag{2}$$

ϕ and S denote, respectively, the porosity and the saturation of the medium and $R(U)$ is the so-called retention function which is taken in our problem to be quadratic for a positive pressure $U(y, \tau)$. All the details for the derivation of the above equations with the boundary conditions can be found in Ref. [6].

We now write the problem we want to solve in one space dimension as follows:

Problem OP. Given T_0 and $g(y)$ which satisfy

$$g(y) \geqslant 0, \quad g'(y) < -1 \quad \text{for} \quad 0 < y \leqslant 1$$

and

$$g'(0) = -1, \quad g(1) = 0.$$

Find the pair $\{U(y, \tau), \quad \bar{s}(\tau)\}$ such that

$$\bar{s}(\tau) > 0, \quad 0 < \tau < T_0, \quad \bar{s}(0) = 1,$$

$$U_{yy}(y, \tau) = \partial R(U)/\partial \tau$$
$$\text{in} \quad \Omega = \{(y, \tau) | 0 < \tau \leqslant T_0, \quad 0 < y < \bar{s}(\tau)\}, \tag{3}$$

$$U_y(0, \tau) = -1, \quad 0 \leqslant \tau \leqslant T_0, \tag{4}$$

$$U[\bar{s}(\tau), \tau] = 0, \quad 0 \leqslant \tau \leqslant T_0, \tag{5}$$

$$U(y, 0) = g(y), \quad 0 \leqslant y \leqslant 1, \tag{6}$$

and in addition

$$\frac{d\bar{s}}{d\tau} + U_y[\bar{s}(\tau), \tau] = -1, \quad 0 \leqslant \tau \leqslant T_0, \tag{7}$$

where

$$R(U) = \begin{cases} U^2 & \text{if} \quad U > 0 \\ 0 & \text{if} \quad U \leqslant 0; \end{cases} \tag{8}$$

(i) $R(U)$ is the retention function which depends on the pressure U, (ii) $\bar{s}(\tau)$ is the free boundary and (iii) $g(y)$ is the initial pressure.

3. VARIATIONAL FORMULATION

The transformed problem

Using the Landau transformation, we introduce the new space variable

$$x = \frac{y}{\bar{s}(\tau)}, \tag{9}$$

while the new time variable $t = t(\tau)$ will be defined by the unique solution to the ordinary differential equation

$$\frac{dt}{d\tau} = \frac{1}{\bar{s}^2(\tau)} \quad \text{with} \quad t(0) = 0. \tag{10}$$

Setting

$$\begin{cases} u(x,t) = U(y,\tau) + y \\ \tilde{s}(\tau) = s(t) \end{cases} \tag{11}$$

and using the new variables, one gets the transformed problem.

Problem TP. Find $\{u(x,t), s(t)\}$ such that

$$u_{xx} - R_t(u_1) = xu_x(1,t)R_x(u_1)$$

$$\text{in} \quad Q = \{(x,t)|0 < x < 1, \quad 0 < t \leqslant T\}, \tag{12}$$

$$u_x(0,t) = 0, \quad 0 < t \leqslant T, \tag{13}$$

$$u(1,t) = s(t), \quad 0 < t \leqslant T, \tag{14}$$

$$u(x,0) = g(x) + x, \quad 0 \leqslant x \leqslant 1, \tag{15}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -su_x(1,t), \quad s(0) = 1, \quad 0 < t \leqslant T, \quad s(t) > 0, \tag{16}$$

where

$$u_1 = u - sx, \quad R_x = \frac{\partial R}{\partial x} \quad \text{and} \quad R_t = \frac{\partial R}{\partial t}$$

and $t = T$ is the corresponding value of $\tau = T_0$

The variational problem

To reach a weak formulation of Problem TP, we make another change of variable. Let

$$v(x,t) = u_x(x,t). \tag{17}$$

For t fixed, we define v in the space

$$\dot{H}^1 = \{w | w \in H^1(0,1) \quad \text{and} \quad w(0) = 0\}.$$

Taking into account equation (14), we obtain the relation

$$u = -\int_x^1 v \,\mathrm{d}x + s(t). \tag{18}$$

Let us choose an arbitrary function $w_x \in \dot{H}^1$, multiply equation (12) by this function and integrate over (0, 1) to obtain

$$\int_0^1 v_x w_x \mathrm{d}x + \int_0^1 R_{xt}(u_1) w \,\mathrm{d}x = v(1) \int_0^1 xR_x(u_1) w_x \mathrm{d}x, \tag{19}$$

where $v(1) = v(1,x)$. The weak formulation of Problem TP now reads as below.

Problem VP. Find $\{v(x,t), s(t)\}$ such that

$$(v_x, w_x) + [R_{xt}(u_1), w] = v(1)[xR_x(u_1), w_x] \forall w \in \dot{H}^1 \quad \text{and} \quad 0 < t \leqslant T \tag{20}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -sv(1,t), \tag{21}$$

with the initial conditions

$$v(x,0) = g'(x) + 1 \tag{22}$$

and

$$s(0) = 1. \tag{23}$$

Here $(.,.)$ denotes the $L_2(0,1)$ scalar product.

Statement of the discretized problem

Using Ritz–Galerkin's method, we sought a solution of the variational problem in a finite dimensional subspace of $\dot{H}^1$, denoted by S_h.

Let Π_h be a subdivision of the interval [0, 1] into $N = 1/h$ equal parts of length h and S_h the space of continuous functions which are piecewise polynomials of degree less than an integer r. We define $\dot{S}_h$ as

$$\dot{S}_h = \{v_h | v_h \in S_h \quad \text{and} \quad v_h(0) = 0\},$$

so that

$$\dot{S}_h \subset \dot{H}^1.$$

The discretized problem now reads as follows.

Problem DP. Find $\{v_h(x, t), s_h(t)\}$ such that

$$(v_{h,x}, w_x) + [R_{xt}(u_{h1}), w_x] = v_h(1)[xR_x(u_{h1}), w_x] \forall w \in \dot{S}_h \quad \text{and} \quad 0 < t \leqslant T \tag{24}$$

and

$$\frac{ds_h}{dt} = -s_h v_h(1), \tag{25}$$

where

$$u_{h1} = u_h - s_h x \tag{26}$$

with the initial conditions

$$v_h(x, 0) = P_h g'(x) + 1, \quad 0 \leqslant x \leqslant 1 \tag{27}$$

and

$$s_h(0) = 1, \tag{28}$$

where P_h is an appropriate projection in the space $\dot{S}_h$.

4. THE NUMERICAL SCHEME

The method of discretization in time consists of the following. Let $\dot{S}_h$ be the space of continuous functions which are a piecewise polynomial of third degree. Divide the interval [0, 1] by the points

$$t_n = nk, \quad n = 0, 1, \ldots, M.$$

The derivative $\partial R_x/\partial t$ in equation (24) is replaced, for $t = t^{n+1}$, by the difference quotient

$$R_{xt}(u_{h1}) = \frac{R_x(u_{h1}^{n+1}) - R_x(u_{h1}^n)}{k}, \tag{29}$$

where

$$u_{h1}^n = u_{h1}(x, nk).$$

Consequently, equation (24) takes the form

$$\frac{1}{k}\{[R_x(u_{h1}^{n+1}), w] - [R_x(u_{h1}^n), w]\} + (v_{h,x}^{n+1}, w_x) = v_h^n(1)[xR_x(u_{h1}^n), w_x],$$

i.e.

$$[R_x(u_{h1}^{n+1}), w] + k(v_{h,x}^{n+1}, w_x) = kv_h^n[xR_x(u_{h1}^n), w_x] + [R_x(u_{h1}^n), w] \forall w \in \dot{S}_h. \tag{30}$$

Similarly, using the backward difference for the time derivative $\partial s_h/\partial t$ and $\partial \tau_h/\partial t$ we obtain, in accordance with equations (25) and (10),

$$s_h^{n+1} = s_h^n\left[1 - \frac{k}{2}(v_h^{n+1}(1) + v_h^n(1) - kv_h^{n+1}(1)\, v_h^n(1))\right] \tag{31}$$

and

$$\tau_h^{n+1} = \tau_h^n + \frac{k}{2}[(s_h^{n+1})^2 + (s_h^n)^2]. \tag{32}$$

Using the bases $\{\psi_i\}_{j=0}^N$ defined by the cubic splines (see Ref. [6]), let us consider the discretized problem and solve it approximately, assuming its approximate solution v_h in the form

$$v_h^{n+1}(x) = \sum_{j=0}^{N} \alpha_j^{n+1}\, \psi_j(x), \tag{33}$$

where the coefficients $\alpha_0^{n+1}, \ldots, \alpha_N^{n+1}$ are to be determined. Using the fact that

$$R_x(u_{h1}^{n+1}) = \frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}} \frac{\partial u_{h1}^{n+1}}{\partial x}.$$

One can write equation (30) in the form

$$\left(\frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}}[v_h^{n+1} - s_h^{n+1}], w\right) + k(v_{h,x}^{n+1}, w_x)$$

$$= kv_h^n(1)\left(x\frac{\partial R(u_{h1}^n)}{\partial u_{h1}^n}[v_h^n - s_h^n], w_x\right) + \left(\frac{\partial R(u_{h1}^n)}{\partial u_{h1}^n}\frac{\partial u_{h1}^n}{\partial x}, w\right) \forall w \in \dot{S}_h. \tag{34}$$

Putting equation (33) into equation (34) and setting $w = \psi_i(x)$, we get

$$\sum_{j=0}^{N} \alpha_j^{n+1} \int_0^1 \left[\psi_i(x)\psi_j(x)\frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}} + k\psi_i'(x)\psi_j'(x)\right]\mathrm{d}x$$

$$+ \frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}}\frac{k}{2}s_h^n\psi_i(1 - kv_h^n)\left(\frac{1}{4}\alpha_{N-1}^{n+1} - \alpha_N^{n+1}\right)$$

$$= \int_0^1 \frac{\partial R(u_{h1}^n)}{\partial u_{h1}^n}(v_h^n - s_h^n)[kv_h^n(1)x\psi_i'(x) + \psi_i(x)]\mathrm{d}x + s_h^n\left[1 - \frac{k}{2}v_h^n(1)\right]$$

$$\times \int_0^1 \frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}}\psi_i(x)\,\mathrm{d}x, \tag{35}$$

$$n = 0, 1, \ldots, M \quad \text{and} \quad i = 0, 1, \ldots, N.$$

Because of the term $\partial R(u_{h1}^{n+1})/\partial u_{h1}^{n+1}$, equation (35) is a nonlinear equation. In order to linearize it, we will use a predictor–corrector method to compute first the term $\partial R(u_{h1}^{n+1})/\partial u_{h1}^{n+1}$ and then with this term known we will solve the linear system (35) for the unknowns $\alpha_0^{n+1}, \ldots, \alpha_N^{n+1}$. The new u_h^{n+1} will be used again to predict the nonlinear term $\partial R(u_{h1}^{n+1})/\partial u_{h1}^{n+1}$, and the process continues until the change in u_h^{n+1} is sufficiently small.

We now describe the predictor–corrector method as follows. In equation (12) replace $R_t(u_{h1})$ by the backward difference at $t = (n+1)k$ to get

$$R_t(u_{h1}) = \frac{R(u_{h1}^{n+1}) - R(u_{h1}^n)}{k}$$

and evaluate the rest of the terms in the equation at the previous time step $t = nk$ to obtain

$$R(u_{h1}^{n+1}) = k[u_{h,xx}^n - xu_{h,x}^n(1)\, R_x(u_{h1}^n)] + R(u_{h1}^n). \tag{36}$$

We now use equations (8) to write, for a positive pressure U,

$$R(u_{h1}^n) = (u_h^n - xs_h^n)^2 \tag{37}$$

and

$$R_x(u_{h1}^n) = 2(u_h^n - xs_h^n)(u_{h,x}^n - s_h^n). \tag{38}$$

Let us denote the r.h.s. of equation (36) by C which, according to equations (18) and (33), takes the form

$$C = k \sum_{j=0}^{N} \alpha_j^n \psi_j'(x) + 2kx\left(\tfrac{1}{4}\alpha_{N-1}^n + \alpha_N^n\right)\left[\sum_{j=0}^{N} \alpha_j^n \int_x^1 \psi_j(x)\,dx + s_h^n(x-1)\right]\left[\sum_{j=0}^{N} \alpha_j^n \psi_j(x) - s_h^n\right] + \left[\sum_{j=0}^{N} \alpha_j^n \int_x^1 \psi_j(x)\,dx + s_h^n(x-1)\right]^2 \quad \text{for} \quad n = 1, 2, \ldots, M \tag{39}$$

and

$$C = kg''(x) - 2xk[g'(1) + 1]g(x)g'(x) + [g(x)]^2 \quad \text{for} \quad n = 0. \tag{40}$$

Given C, we can now compute the term

$$\frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}} = 2u_{h1}^{n+1}$$

which, according to equations (36) and (8), takes the form

$$\frac{\partial R(u_{h1}^{n+1})}{\partial u_{h1}^{n+1}} = 2\sqrt{C}, \quad n = 0, 1, \ldots, M,$$

where C is given by equations (37) and (38).

By solving the system (35) we obtain the values of v_h^{n+1} at each point $[ih, (n+1)k]$ $(i = 0, 1, \ldots, N)$, and at the same time calculate s_h^{n+1} and τ_h^{n+1}. Having obtained the values of $v_h(x, t)$ at the points $[ih, (n+1)k]$ $(i = 0, 1, \ldots, N)$ and the values of s_h and τ_h at the points $(n+1)k$, we use equation (18) to obtain $u_h(x, t)$.

Finally, the values of $U_h(y_h, \tau_h)$ are obtained at the points $y_h = ihs_h^{n+1}$ $(i = 0, 1, \ldots, N)$ and $\tau_h = \tau_h^{n+1}$ using the relation

$$U_h(y_h, \tau_h) = u_h[ih, (n+1)k] - ihs_h^{n+1}, \quad i = 0, 1, \ldots, N.$$

We now present a numerical result of the application of the present scheme to the following model problem:

$$U_{yy} - \partial U^2/\partial\tau = 0, \quad 0 < y < \tilde{s}(\tau), \quad 0 < \tau \leqslant 0.62;$$
$$U_y(0, \tau) = -1, \quad 0 \leqslant \tau \leqslant 0.62;$$
$$U(y, 0) = -\tfrac{1}{2}y^3 - \tfrac{1}{2}y^2 - y + 2, \quad 0 \leqslant y \leqslant 1;$$
$$U[\tilde{s}(\tau), \tau] = 0, \quad 0 \leqslant \tau \leqslant 0.62;$$
$$U_y[\tilde{s}(\tau), \tau] = -d\tilde{s}/d\tau - 1, \quad 0 \leqslant \tau \leqslant 0.62;$$

and

$$\tilde{s}(0) = 1.$$

5. NUMERICAL RESULTS

Here we report the results obtained using the above algorithm to solve our model problem.

We used the cubic splines as the basis functions because they belong to the space $\dot{S}_h$ of continuous functions which are a piecewise polynomial of third degree. Since we used an explicit form of the difference equation (36) in the predictor–corrector method, we chose $k = 0.004$ so that $k/h^2 < 1/2$ in order to get convergence. For the integration we used an exact formula, since v is piecewise cubic, and we solved the system of equations (35) by using Gaussian elimination. The error criterion for smallness of change in u_h^{n+1} was $5 \cdot 10^{-4}$ and the number of iterations involved per time step was at most five (typically four).

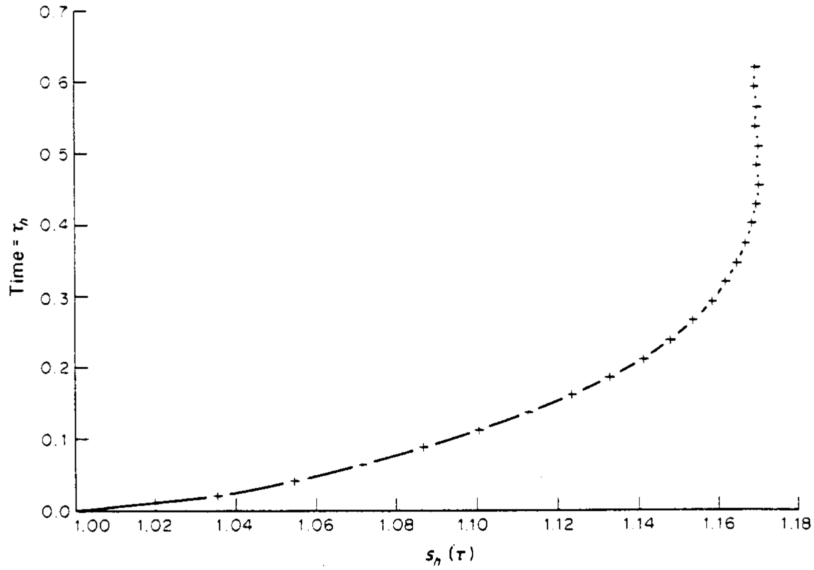

Fig. 1. Free-boundary curve.

The actual computation was carried out using $k = 0.004$, $h = 1/11$, $N = 11$ and $M = 50$. Figure 1 represents the curve of the free boundary $s_h(\tau)$.

Figures 2 and 3 represent the solution $U_h(y_h, \tau_h)$ of Problem OP at different times τ_h.

From Fig. 1 one can see that $s_h(\tau)$ tends to approach an asymptote $s_h(\tau_h) = 1.17$ which shows that at time $\tau_h = 0.46$ a certain residual saturation is reached, One can see also from Fig. 3 that at time $\tau_h = 0.54$ the pressure becomes independent of time. Consequently, from that time the pressure would depend only on the depth. This is expected since no water is allowed into the system and one would expect the downward migration to come to an end.

Since no exact results are known, it is impossible for us to make any statements about the accuracy of our results. However, they are in good agreement with our physical expectations.

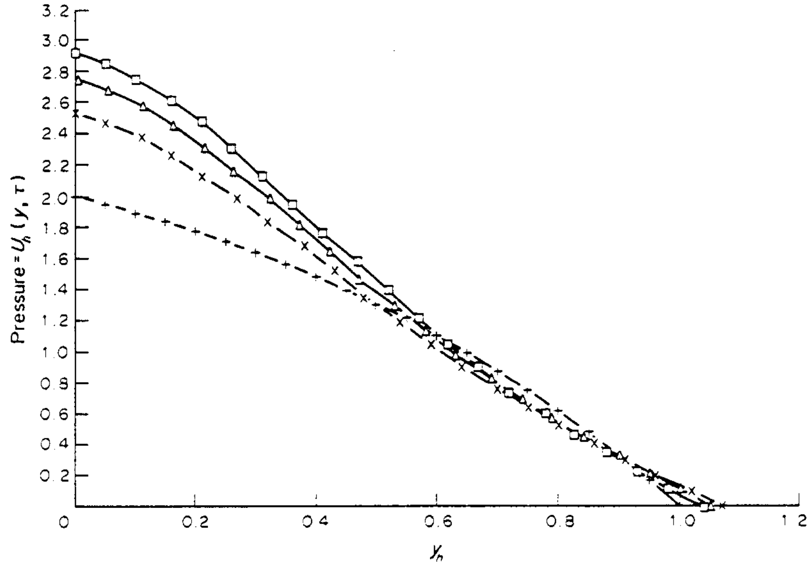

Fig. 2. Pressure at different time steps: $-$, $\tau_h = 0$; $\square$, $\tau_h = 0.02$; $\triangle$, $\tau_h = 0.04$; $\times$, $\tau_h = 0.07$.

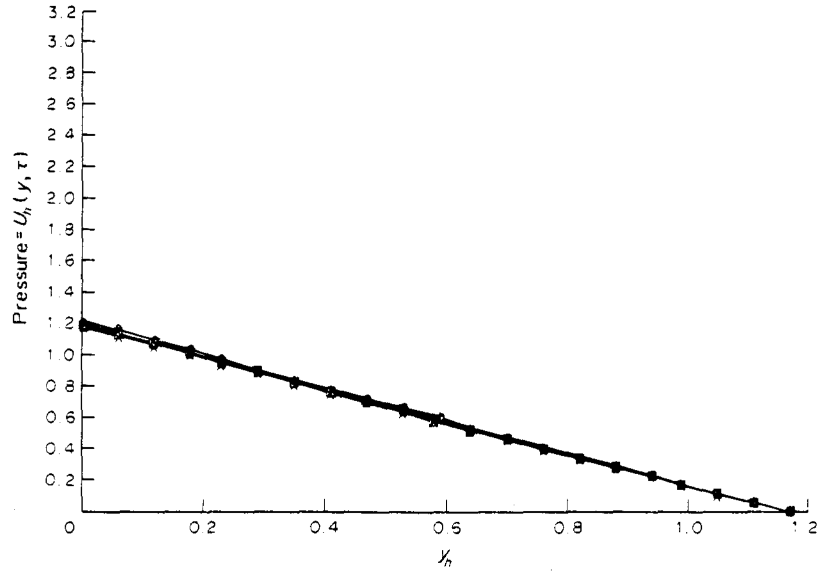

Fig. 3. Pressure at different time steps: +, $\tau_h = 0.51$; □, $\tau_h = 0.54$; △, $\tau_h = 0.57$; ×, $\tau_h = 0.59$; ○, $\tau_h = 0.62$.

6. CONCLUSION

The main advantage of the boundary-fixing method to solve free-boundary problems is to avoid the difficulties arising from having an unknown domain. The main disadvantage is the highly nonlinear structure of the transformed differential equations. Since the solution of the finite dimensional nonlinear system of equations resulting from the discretization of the differential equations is one of the critical phases in the application of this method [7], the transformation method may require a good initial guess for the solution of the problem. By using a predictor–corrector method in this paper we have avoided this difficulty, since at each time step the previous solution is used and then corrected.

The numerical method developed above can be extended to solve the problem in two or three dimensions, since little attention has been paid to date to the boundary-fixing method in several space dimensions. One can also use it to solve the same problem for various forms of the retention function.

REFERENCES

1. G. H. Meyer, The numerical solution of multi dimensional Stefan problems. In *A Survey—in Moving Boundary Problems* (Edited by D. G. Wilson, A. D. Solomon and P. T. Boggs). Academic Press, New York (1978).
2. R. M. Furzland, A survey of the formulation and solution of free and moving boundary (Stefan) problems. Brunel Univ., Bristol, Avon.
3. E. DiBenedetto, Interior and boundary regularity for a class of free boundary problems. In *Research Notes in Mathematics*; *Free Boundary Problems*: *Theory and Applications*, Vol. 2 (Edited by A. Fasano and M. Primicerio), pp. 383–396 (1979).
4. R. J. Lenhard, Effects of clay–water interactions on water retention in porous media. Ph.D. Thesis, Oregon State Univ., Corvallis, Oreg. (1984).
5. W. Fulk, R. B. Guenther and E. L. Roetman, Equation of motion and continuity for fluid flow in porous medium. *Acta mech.* **12,** 121–129 (1971).
6. A. Kharab, A nonlinear free boundary value problem. Ph.D. Thesis, Oregon State Univ., Corvallis, Oreg. (1984).
7. G. H. Meyer, *Initial Value Methods for Boundary Value Problems*. Academic Press, New York (1973).